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Photon mass and planetary magnetic fields

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Abstract. The existence of a photon mass implies additional components of static magnetic fields. Schrödinger's expression for a dipole field is here extended to a complete spherical harmonic analysis of a static planetary field. Additional components of the magnetic field are identified which are due to a photon mass. The magnitude of the most important of these new terms is evaluated for the geomagnetic field and is found to be of the same order as Schrödinger's apparent external field.

1. Introduction

There has been considerable interest in recent years in establishing an upper limit on the mass of the photon. A modification of Maxwell's equations which is consistent with a photon mass is the Proca equations (in four-tensor form)

$$F^{ij}{}_{;j} - \mu^2 A^i = 4\pi j^i \tag{1}$$

$$A^i{}_{;i} = 0, \tag{2}$$

where μ is related to the photon mass m by $\mu = mc/\hbar$. Schrödinger's (1943) expression for a magnetic dipole of moment $\boldsymbol{m}$:

$$\boldsymbol{H} = (\mathrm{e}^{-\mu r}/r^3)\{[1+\mu r+\tfrac{1}{3}(\mu r)^2][3(\boldsymbol{m}\,.\,\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}] - \tfrac{2}{3}(\mu r)^2\boldsymbol{m}\}, \tag{3}$$

was applied by Goldhaber and Nieto (1968) to Cain's (1966) analysis of the geomagnetic field. This led to a limit on μ of $10^{-10}\,\mathrm{cm}^{-1}$. A similar analysis of Jupiter's magnetic field leads to a limit of $2\times10^{-11}\,\mathrm{cm}^{-1}$ (Davis *et al* 1975).

There have been other attempts at establishing upper limits on μ which rely upon astrophysical observations, for example, Barnes and Scargle (1975). While these methods seem more promising, the static magnetic field methods of Davis *et al* are probably the most rigorous. It is therefore of interest to examine further the relationship between photon mass and static magnetic fields. In this paper a complete spherical harmonic analysis of static planetary fields is given extending Schrödinger's expression (equation (3)) to all orders of the multipole moments. Additional terms in the magnetic field due to a photon mass will be identified and an estimate of the most important of these will be made for the geomagnetic field.

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2. Solution of Proca's equations for a static planetary field

Proca's equations for a static magnetic field become (in three-dimensional form)

$$(\nabla^2-\mu^2)\boldsymbol{A}=-4\pi\boldsymbol{j} \tag{4}$$

$$\operatorname{div}\boldsymbol{A}=0 \tag{5}$$

$$\boldsymbol{H}=\operatorname{curl}\boldsymbol{A}. \tag{6}$$

One may solve for the vector $\boldsymbol{A}$ by expanding it in terms of the orthonormal vector spherical harmonics $\boldsymbol{Y}^m_{n,n_1}$ whose components in spherical polar coordinates are given by

$$[(n+1)(2n+1)]^{1/2}\boldsymbol{Y}^m_{n,n+1}=-(n+1)Y^m_n\hat{\boldsymbol{r}}+\frac{\partial Y^m_n}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{1}{\sin\theta}\frac{\partial Y^m_n}{\partial\phi}\hat{\boldsymbol{\phi}} \tag{7}$$

$$[n(n+1)]^{1/2}\boldsymbol{Y}^m_{n,n}=\frac{\mathrm{i}}{\sin\theta}\frac{\partial Y^m_n}{\partial\phi}-\mathrm{i}\frac{\partial Y^m_n}{\partial\theta}\hat{\boldsymbol{\phi}} \tag{8}$$

$$[n(2n+1)]^{1/2}\boldsymbol{Y}^m_{n,n-1}=nY^m_n\hat{\boldsymbol{r}}+\frac{\partial Y^m_n}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{1}{\sin\theta}\frac{\partial Y^m_n}{\partial\phi}\hat{\boldsymbol{\phi}} \tag{9}$$

where the Y^m_n are normalized surface harmonics. For a complete account of the properties of these harmonics the reader is referred to Winch and James (1973), Winch (1974) and James (1974).

In solving equations (4) and (5) we put

$$\boldsymbol{A}=\int\frac{\mathrm{e}^{-\mu R}}{R}\boldsymbol{j}_0\,\mathrm{d}v_0 \tag{10}$$

where the integral is over the planetary volume assumed to be a sphere of radius a and the zero subscript refers to a source point. We expand $\boldsymbol{j}$ in terms of the $\boldsymbol{Y}^m_{n,n_1}$

$$\boldsymbol{j}(\boldsymbol{r}_0)=\sum_{n=0}^{\infty}\sum_{\nu=-1}^{+1}\sum_{m=-n}^{+n}J^m_{n,n+\nu}(r_0)\boldsymbol{Y}^m_{n,n+\nu}(\theta_0,\phi_0). \tag{11}$$

This gives

$$\boldsymbol{A}=\sum_{n=0}^{\infty}\sum_{\nu=-1}^{+1}\sum_{m=-n}^{+n}\tilde{J}^m_{n,n+\nu}k_{n+\nu}(r)\boldsymbol{Y}^m_{n,n+\nu}(\theta,\phi), \tag{12}$$

the $k_n(r)$ being modified spherical Bessel functions of the third kind

$$k_n(r)=\frac{\mathrm{e}^{-\mu r}}{r^{n+1}}\left(1+\sum_{p=1}^{n}\frac{n!}{(2n)!}\frac{(2n-p)!}{(n-p)!p!}(2\mu r)^p\right).$$

The current moments $\tilde{J}^m_{n,n+\nu}$ appearing in (12) are given by

$$\tilde{J}^m_{n,n+\nu}=\frac{4\pi}{2n+1}\int_0^a r_0^2 i_{n+\nu}(r_0)J^m_{n,n+\nu}(r_0)\,\mathrm{d}r_0, \tag{13}$$

where $i_n(r)$ is a modified spherical Bessel function of the first kind normalized so that

$$i_n(r)=r^n[1+\mathrm{O}(\mu r)^2].$$

The fact that $\boldsymbol{A}$ has no divergence leads to the condition on the current moments

$$\tilde{J}^m_{n,n-1}=\frac{\mu^2}{4n-1}\left(\frac{n+1}{n}\right)\tilde{J}^m_{n,n+1}, \tag{14}$$

so that for each n, m there are only two independent such moments. The expression for $\boldsymbol{H}=\operatorname{curl}\boldsymbol{A}$ is

$$\begin{aligned}\boldsymbol{H}=-\mathrm{i}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n}&\mu^2\left(\frac{n+1}{2n+1}\right)^{1/2}\frac{1}{2n-1}(a^n k_{n-1})\tilde{J}^m_{n,n}\boldsymbol{Y}^m_{n,n-1}\\&+\frac{\mu^2}{n^{1/2}(2n+1)^{1/2}}(a^{n+1}k_n)\tilde{J}^m_{n,n+1}\boldsymbol{Y}^m_{n,n}\\&+n^{1/2}(2n+1)^{1/2}(a^{n+2}k_{n+1})\frac{\tilde{J}^m_{n,n}}{a^{n+2}}\boldsymbol{Y}^m_{n,n+1}.\end{aligned} \tag{15}$$

The k_n have been normalized by a^{n+1} so that they are of order unity on the surface of the planet.

By comparison of (15) with the expression for a planetary magnetic field obtained from Maxwell's equations (cf Winch and James 1973) the coefficients of the $\boldsymbol{Y}^m_{n,n-1}$ and $\boldsymbol{Y}^m_{n,n}$ terms which are proportional to μ^2 reveal the existence of *apparent* external and non-potential fields. (An external field is one caused by external currents, a non-potential field is that due to surface to atmosphere currents.) On the basis of Maxwell theory we would associate the coefficients of the $\boldsymbol{Y}^m_{n,n+1}$ terms with the internal field. Since the $\boldsymbol{Y}^m_{n,n-1}$ and $\boldsymbol{Y}^m_{n,n+1}$ terms are dependent upon the same current moment $\tilde{J}^m_{n,n}$ the coefficients of the apparent external field are proportional to those of the internal field. In terms of the usual Gauss–Schmidt coefficients

$$\left\{\begin{matrix} g^m_{n,\mathrm{e}}/g^m_{n,\mathrm{i}} \\ h^m_{n,\mathrm{e}}/h^m_{n,\mathrm{i}}\end{matrix}\right\}=(\mu a)^2\frac{n+1}{n(4n^2-1)}, \tag{16}$$

to the first order in $(\mu a)^2$. (The subscripts e and i refer to external and internal.)

Also by comparison with the Maxwell field we find the (complex) apparent non-potential coefficients to be (cf Winch and James 1973 for the definition of J^m_n)

$$J^m_n=-\frac{(\mu a)^2}{n(2n+1)}\frac{\tilde{J}^m_{n,n+1}}{a^{n+3}}, \tag{17}$$

to the first order in $(\mu a)^2$. The current moments $\tilde{J}^m_{n,n+1}$, are not *a priori* related to the $\tilde{J}^m_{n,n}$ and so the size of the apparent non-potential field cannot be estimated immediately from the coefficients of the internal field.

3. Estimate of apparent non-potential field for the earth

In order to estimate the $\tilde{J}^m_{n,n+1}$ current moments use was made of a geomagnetic dynamo model of Peheris *et al* (1973).

The above authors expand their field in a series of poloidal and toroidal terms with corresponding coefficients $S^m_n(r)$ and $T^m_n(r)$ respectively, where r is a radial coordinate normalized to the value 1 at the core radius. Using

$$4\pi\boldsymbol{j}=\operatorname{curl}\boldsymbol{H}, \tag{18}$$

it may be shown that

$$\tilde{J}^m_{n,n+1} = -n\left(\frac{2n+3}{2n+1}\right)\left(\frac{n+1}{2n+1}\right)^{1/2} b^{n+3}\int_0^1 r^{n+1}T^m_n\,\mathrm{d}r, \tag{19}$$

and

$$\tilde{J}^m_{n,n} = \frac{\mathrm{i}n^{1/2}(n+1)^{1/2}}{2n+1}b^{n+1}S^m_n(1), \tag{20}$$

where b is the core radius. The ratio of the apparent external to internal coefficients is therefore given by

$$\frac{J^m_n}{G^m_{n,\mathrm{i}}} = (\mu a)^2\frac{2n+3}{n(2n+1)^2}\left(\frac{n+1}{2n+1}\right)^{1/2}\frac{b}{a}\frac{\int_0^1 r^{n+1}T^m_n\,\mathrm{d}r}{S^m_n(1)}. \tag{21}$$

The $G^m_{n,\mathrm{i}}$ are complex combinations of the usual internal Gauss–Schmidt coefficients $g^m_{n,\mathrm{i}}$ and $h^m_{n,\mathrm{i}}$ (cf Winch and James 1973).

In their model Peheris *et al* take

$$S^0_1 = 1, \qquad S^1_1 = S^{-1}_1 = T^{-1}_1 = T^1_1 = 0, \tag{22}$$

the magnetic field being expressed in dipole coordinates. It thus follows from (21) and (22) that

$$J^1_1 = J^{-1}_1 = 0. \tag{23}$$

A simple numerical integration gives

$$J^0_1 = -0{\cdot}66(\mu a)^2 G^0_{1,\mathrm{i}}, \tag{24}$$

J^0_1 gives the greatest contribution to the apparent non-potential field. The magnetic field associated with this term

$$H' = \mathrm{i}3^{1/2}J^0_1 a^2 k_1(r)\boldsymbol{Y}^0_{1,1}, \tag{25}$$

may be written as

$$\boldsymbol{H}' = 0{\cdot}81(\mu a)^2\,\mathrm{e}^{-\mu r}(1+\mu r)\left(\frac{a}{r}\right)^2\frac{\boldsymbol{m}}{a^3}\times\hat{\boldsymbol{r}}, \tag{26}$$

where $\boldsymbol{m}$ is the geomagnetic dipole moment and where we have taken into account equation (24). The complete $n = 1$ geomagnetic field is therefore

$$\boldsymbol{H} = (\mathrm{e}^{-\mu r}/r^3)\{[1+\mu r+\tfrac{1}{3}(\mu r)^2][3(\boldsymbol{m}\,.\,\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}-\boldsymbol{m}]-\tfrac{2}{3}(\mu r)^2\boldsymbol{m}\}$$
$$+0{\cdot}81(\mu a)^2\,\mathrm{e}^{-\mu r}(1+\mu r)\left(\frac{a}{r}\right)^2\frac{\boldsymbol{m}}{a^3}\times\hat{\boldsymbol{r}}, \tag{27}$$

where it should be understood the last term is only approximate.

4. Discussion and conclusions

Two main results have been presented here, the expression for the nth order multipole coefficient of the apparent external field

$$\left\{\begin{matrix} g^m_{n,\mathrm{e}}/g^m_{n,\mathrm{i}} \\ h^m_{n,\mathrm{e}}/h^m_{n,\mathrm{i}} \end{matrix}\right\} = (\mu a)^2\frac{n+1}{n(4n^2-1)}, \tag{28}$$

and the expression for the coefficients of the apparent non-potential field

$$J_n^m = (\mu a)^2 \frac{2n+3}{(2n+1)^2}\left(\frac{n+1}{2n+1}\right)^{1/2}\left(\frac{b}{a}\right)^{n+3}\int_0^1 r^{n+1} T_n^m \, dr. \tag{29}$$

Unfortunately in the case of the earth, the apparent external quadrupole coefficients would be too small to be measured. Substituting the Davis–Goldhaber–Nieto limit of 2×10^{-11} cm^{-1} into equation (28) one finds that $g_{n,e}^m$ and $h_{n,e}^m$ are all of the order of $0{\cdot}1\gamma$ (10^{-6} Oe) or less. A resolution of the geomagnetic field of this order is unobtainable at present. The values obtained by Peheris *et al* for the higher order internal toroidal fields indicate that the higher order apparent non-potential coefficients would also be too small to measure.

However, the results obtained here for the apparent non-potential field could be used to further reduce upper estimates of the photon mass by the use of data from planetary magnetic fields. For the geomagnetic field we have the approximate non-potential terms (cf equation (27)) which are of the same order of magnitude as Schrödinger's apparent external field. Estimates of the two fields in the case of the earth could feasibly reduce the present upper limit on the photon mass. Unfortunately at the time of writing there do not appear to be any accurate spherical harmonic analyses of the geomagnetic field which estimate the non-potential terms. As far as other planetary magnetic fields are concerned it is not possible at the present time to make an approximate evaluation of the apparent non-potential field. However, the results for the geomagnetic field indicate that the apparent non-potential terms should be of the same order of magnitude as the apparent external terms, and hence inclusion of the former in a spherical harmonic analysis of planetary magnetic fields is demanded. Exclusion of such terms would affect the estimates of the apparent external terms and these could mask the tendency of the external terms to follow the relations expressed in equation (28).

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